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An infinite dimensional version of the intermediate value theorem

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Abstract. Let $\mathfrak{f} = I - k$ be a compact vector field of class C^1 on a real Hilbert space $\mathbb{H}$. In the spirit of Bolzano's Theorem on the existence of zeros in a bounded real interval, as well as the extensions due to Cauchy (in $\mathbb{R}^2$) and Kronecker (in $\mathbb{R}^k$), we prove an existence result for the zeros of $\mathfrak{f}$ in the open unit ball $\mathbb{B}$ of $\mathbb{H}$. Similarly to the classical finite dimensional results, the existence of zeros is deduced exclusively from the restriction $\mathfrak{f}|_{\mathbb{S}}$ of $\mathfrak{f}$ to the boundary $\mathbb{S}$ of $\mathbb{B}$. As an extension of this, but not as a consequence, we obtain as well an Intermediate Value Theorem whose statement needs the topological degree. Such a result implies the following easily comprehensible, nontrivial, generalization of the classical Intermediate Value Theorem: *If a half-line with extreme $q \notin \mathfrak{f}(\mathbb{S})$ intersects transversally the function $\mathfrak{f}|_{\mathbb{S}}$ for only one point* of $\mathbb{S}$, *then any value of the connected component of $\mathbb{H} \backslash \mathfrak{f}(\mathbb{S})$ containing q is assumed by $\mathfrak{f}$ in $\mathbb{B}$. In particular, such a component is bounded.*

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1. Introduction

Let $\mathbb{H}$ be a real Hilbert space. Denote by $\mathbb{B}$ the open unit ball of $\mathbb{H}$ and by $\overline{\mathbb{B}}$ its closure, also called the *unit disk* of $\mathbb{H}$. The boundary $\partial\mathbb{B}$ of $\mathbb{B}$ is the unit sphere of $\mathbb{H}$, hereafter denoted by $\mathbb{S}$.

Let $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ be a compact vector field; namely, a map of the type $I - k$, where I is the identity of $\mathbb{H}$ and k is a compact map, meaning that k is

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continuous and sends bounded sets into relatively compact sets. This implies that $\mathfrak{f}(\mathbb{S})$ is a bounded closed subset of $\mathbb{H}$.

In this paper, among other results, we obtain the following statement (Theorem 6.9), which extends, to the infinite dimensional setting, the classical one-dimensional Intermediate Value Theorem:

Assume that $\mathfrak{f}$ *is of class* C^1, *and let* $\Lambda \subset \mathbb{H}$ *be a half-line with extreme* $q \notin \mathfrak{f}(\mathbb{S})$. *If the intersection* $\mathfrak{f}(\mathbb{S}) \cap \Lambda$ *is transverse and is the image under* $\mathfrak{f}$ *of an odd number of points of* $\mathbb{S}$, *then the connected component of* $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$ *containing* q *is a bounded open subset of* $\mathfrak{f}(\mathbb{B})$.

As we shall see, such an easily comprehensible statement is a consequence of our main result (Theorem 6.5), whose formulation requires a notion of degree for special maps between finite or infinite dimensional manifolds introduced in [6], hereafter called *bf-degree* to distinguish it from other classical degrees, such as the Brouwer degree, *Br-degree*, and the Leray–Schauder degree, *LS-degree* (see [5,7,8,25] for additional details).

We shall denote by $\mathfrak{r}$ the radial retraction of $\mathbb{H}\backslash\{0\}$ onto the unit sphere $\mathbb{S}$. That is, $\mathfrak{r}$ is the smooth map $p \mapsto p/\|p\|$. By the *boundary map* (of $\mathfrak{f}$) we mean the restriction $\mathfrak{f}|_{\mathbb{S}} \colon \mathbb{S} \to \mathbb{H}$ of $\mathfrak{f}$ to the boundary $\mathbb{S}$ of $\mathbb{B}$. If $0 \notin \mathfrak{f}(\mathbb{S})$, an important map makes sense: the *boundary self-map* $\mathfrak{f}^\partial \colon \mathbb{S} \to \mathbb{S}$ of $\mathfrak{f}$, given by the composition $\mathfrak{r} \circ \mathfrak{f}|_{\mathbb{S}}$ of the boundary map $\mathfrak{f}|_{\mathbb{S}}$ with the radial retraction $\mathfrak{r}$.

Observe that, if $\mathbb{H}$ is finite dimensional and $0 \notin \mathfrak{f}(\mathbb{S})$, then the Brouwer degree, $\deg_{Br}(\mathfrak{f}^\partial)$, of the boundary self-map $\mathfrak{f}^\partial$ is well defined; this being a special case of the degree for maps between oriented, finite dimensional, compact, connected, real manifolds (see, for example, [15,19]). For a self-map, such as $\mathfrak{f}^\partial$, one assumes that the orientations of domain and codomain are the same.

We recall that, when $\dim \mathbb{H} = 2$, the integer $\deg_{Br}(\mathfrak{f}^\partial)$ is called the *winding number (around the origin) of the closed curve* $\mathfrak{f}|_{\mathbb{S}} \colon \mathbb{S} \to \mathbb{H}$. For this reason, even when the (finite) dimension of $\mathbb{H}$ is higher than 2 (and $0 \notin \mathfrak{f}(\mathbb{S})$), it is folklore to say that $\deg_{Br}(\mathfrak{f}^\partial)$ is the *winding number (around the origin) of the map* $\mathfrak{f}|_{\mathbb{S}}$.

Actually, still in the case when $\dim \mathbb{H} < \infty$ and $0 \notin \mathfrak{f}(\mathbb{S})$, $\deg_{Br}(\mathfrak{f}^\partial)$ may be considered as a modern reformulation of the *index of the boundary map* $\mathfrak{f}|_{\mathbb{S}}$ introduced by L. Kronecker in [17], hereafter denoted by $I(\mathfrak{f}|_{\mathbb{S}})$. Such an integer, when different from zero, ensures the existence of at least one zero of the map $\mathfrak{f}$ in $\mathbb{B}$ (see, for example, [20]; see also [12] for details and interesting historical notes).

Our main topological tool is the bf-degree, denoted here $\deg_{bf}(\cdot)$ to distinguish it from the Brouwer degree $\deg_{Br}(\cdot)$. Since, in the finite dimensional case, the absolute values of these two degrees coincide, $|\deg_{bf}(\mathfrak{f}^\partial)|$ may be regarded as an extension of $|I(\mathfrak{f}|_{\mathbb{S}})|$ to the case $\dim \mathbb{H} = \infty$.

Given any $q \notin \mathfrak{f}(\mathbb{S})$, we prove that, if $|\deg_{bf}((\mathfrak{f} - q)^\partial)| \neq 0$, then the connected component of $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$ containing q is open and included in the bounded set $\mathfrak{f}(\mathbb{B})$. Such a result (Theorem 6.5) is our infinite dimensional version of the Intermediate Value Theorem.

As we shall see, an easily understandable hypothesis implying the condition $|\deg_{bf}((\mathfrak{f}-q)^{\partial})| \neq 0$ is the following: *There exists a half-line with extreme* q *whose intersection with* $\mathfrak{f}(\mathbb{S})$ *is transverse and its preimage under* $\mathfrak{f}|_{\mathbb{S}}$ *consists of an odd number of points.*

A final example will show that the converse implication is not true.

It is worth pointing out that many authors addressed the problem of defining an integer-valued degree for Fredholm maps; see, e.g., [4,21,25] for a comprehensive discussion. Among them we cite Fitzpatrick, Pejsachowicz and Rabier, who defined in [13] a notion of degree for C^2 Fredholm maps between real Banach manifolds, based on a concept of orientation for this class of maps; such a degree has been extended to the C^1 case in [22]. This notion of orientation is different from the one that we follow here, introduced by the first and third author in [6]. We think that a result analogous to Theorem 6.5 holds true in this context; however we are not able to prove it because of some technical difficulties.

2. Preliminaries

Here we expose some notation and preliminaries that we will need later.

Recall that a continuous map between metrizable spaces is said to be *proper* if the pre-image of any compact set is a compact set. It is easy to check that proper maps are *closed*, in the sense that the image of any closed set is a closed set.

Let, hereafter, $\mathbb{H}$ and $\mathbb{K}$ be two real Hilbert spaces. By $\mathcal{L}(\mathbb{H},\mathbb{K})$ we will denote the Banach space of the bounded linear operators from $\mathbb{H}$ into $\mathbb{K}$, endowed with the usual operator norm. For simplicity, the notation $\mathcal{L}(\mathbb{H})$ stands for $\mathcal{L}(\mathbb{H},\mathbb{H})$. By $\mathrm{Iso}(\mathbb{H},\mathbb{K})$ we shall mean the open subset of $\mathcal{L}(\mathbb{H},\mathbb{K})$ of the invertible operators, and we will write $\mathrm{GL}(\mathbb{H})$ instead of $\mathrm{Iso}(\mathbb{H},\mathbb{H})$.

We recall that an operator $L \in \mathcal{L}(\mathbb{H},\mathbb{K})$ is said to be *Fredholm* (see e.g. [24]) if both its kernel, $\operatorname{Ker} L$, and its cokernel, $\operatorname{coKer} L = \mathbb{K}/L(\mathbb{H})$, are finite dimensional. In this case its *index* is the integer

$$\operatorname{ind} L = \dim(\operatorname{Ker} L) - \dim(\operatorname{coKer} L).$$

Obviously, if $L \in \mathcal{L}(\mathbb{H},\mathbb{K})$ is invertible, then it is Fredholm of index 0. Moreover, any operator in $\mathcal{L}(\mathbb{R}^k,\mathbb{R}^s)$ is Fredholm with index $k-s$.

By $\Phi(\mathbb{H},\mathbb{K})$ we denote the subset of $\mathcal{L}(\mathbb{H},\mathbb{K})$ of the Fredholm operators. Given $n \in \mathbb{Z}$, $\Phi_n(\mathbb{H},\mathbb{K})$ stands for $\{L \in \Phi(\mathbb{H},\mathbb{K}) : \operatorname{ind} L = n\}$. In particular, $\Phi(\mathbb{H})$ and $\Phi_n(\mathbb{H})$ designate $\Phi(\mathbb{H},\mathbb{H})$ and $\Phi_n(\mathbb{H},\mathbb{H})$, respectively.

Proposition 2.1. *Here are some important properties of the Fredholm operators:*

1. *if* $L \in \Phi(\mathbb{H},\mathbb{K})$*, then the image of* L *is closed in* $\mathbb{K}$*;*
2. *the composition of Fredholm operators is Fredholm and its index is the sum of the indices of the composite operators;*
3. *if* $L \in \Phi(\mathbb{H},\mathbb{K})$*, then* L *is proper on any bounded and closed subset of* $\mathbb{H}$*;*
4. *for any* $n \in \mathbb{Z}$*, the set* $\Phi_n(\mathbb{H},\mathbb{K})$ *is open in* $\mathcal{L}(\mathbb{H},\mathbb{K})$*;*

5. *if* $L \in \Phi_n(\mathbb{H}, \mathbb{K})$ *and* $K \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ *is compact, then* $L + K \in \Phi_n(\mathbb{H}, \mathbb{K})$.

An useful consequence of Property 2 is the following:

- If $L \in \Phi_n(\mathbb{H}, \mathbb{K})$ and $s \in \mathbb{N}$, then the restriction of L to an s-codimensional closed subspace of $\mathbb{H}$ is Fredholm of index $n - s$.

Let $f : W \to \mathbb{K}$ be a C^1 map defined on an open subset of $\mathbb{H}$. Recall that f is said to be *Fredholm of index* $n \in \mathbb{Z}$ if, for all $p \in W$, the Fréchet differential df_p of f at p belongs to $\Phi_n(\mathbb{H}, \mathbb{K})$. In the sequel we will say that f is a Φ_n-*map*.

Hereafter, for short, by a *manifold* we shall mean a smooth boundaryless differentiable (finite or infinite dimensional, locally closed) manifold embedded in a real Hilbert space. Therefore, in what follows, any manifold has an induced Riemannian structure. Notice that the sphere $\mathbb{S}$ is a one-codimensional manifold of $\mathbb{H}$.

If $\mathcal{M}$ is a manifold embedded in a Hilbert space $\hat{\mathbb{H}}$ and $p \in \mathcal{M}$, the tangent space of $\mathcal{M}$ at p, denoted by $T_p\mathcal{M}$, will be identified with a closed subspace of $\hat{\mathbb{H}}$. In fact, one may regard any tangent vector $\dot{p} \in T_p\mathcal{M}$ as the derivative $\gamma'(0) \in \hat{\mathbb{H}}$ of a C^1 curve $\gamma : (-1, 1) \to \mathcal{M}$ such that $\gamma(0) = p$.

As for maps between Hilbert spaces, a C^1 map $f : \mathcal{M} \to \mathcal{N}$ between two manifolds is *Fredholm of index* n (see [23]) if so is the differential $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$, for any $p \in \mathcal{M}$. Such a map will be called a Φ_n-*map (between manifolds)*.

Given a map $f : \mathcal{M} \to \mathcal{N}$, one usually calls, respectively, *points* and *values* the elements in the *domain* $\mathcal{M}$ and the *codomain* $\mathcal{N}$ of f.

If $f : \mathcal{M} \to \mathcal{N}$ is C^1, an element $p \in \mathcal{M}$ is said to be a *regular point* if the differential $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ is surjective, otherwise p is a *critical point*. An element $p \in \mathcal{N}$ is a *critical value* if its pre-image $f^{-1}(q)$ contains at least one critical point, otherwise p is a *regular value*.

The celebrated Sard's Lemma (see e.g. [15]) implies that, if $\mathcal{M}$ and $\mathcal{N}$ have the same finite dimension and f is C^1, then the set of regular values is dense in $\mathcal{N}$. Moreover (see also [23]), by a finite dimensional reduction argument, one can show that the same assertion holds true even when f is a proper Φ_0-map acting between infinite dimensional manifolds.

3. Algebraic and topological orientations

This section is devoted to a summary regarding the notion of algebraic orientation for linear Fredholm operators of index 0 between Hilbert spaces, as well as to the concept of topological orientation for nonlinear Fredholm maps of index 0 between manifolds. The reader can see [6,7].

3.1. Special linear operators and algebraic orientation

By $\mathcal{F}(\mathbb{H}, \mathbb{K})$, or simply by $\mathcal{F}(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$, we denote the (not necessarily closed) vector subspace of $\mathcal{L}(\mathbb{H}, \mathbb{K})$ of the operators with finite dimensional image.

Unless otherwise stated, with the symbol I we mean the identity operator acting on any vector space.

If $L \in \mathcal{L}(\mathbb{H})$ has the property that $I - L \in \mathcal{F}(\mathbb{H})$, we shall say that L is a *det-admissible operator*. The symbol $\mathcal{A}(\mathbb{H})$ will stand for the affine subspace $I - \mathcal{F}(\mathbb{H})$ of $\mathcal{L}(\mathbb{H})$ of the det-admissible operators. Obviously, if $\mathbb{H}$ is finite dimensional, then $\mathcal{A}(\mathbb{H}) = \mathcal{L}(\mathbb{H})$.

Since the identity on $\mathbb{H}$ is Fredholm of index 0, from Property 5 of Proposition 2.1 one gets that $\mathcal{A}(\mathbb{H})$ is a subset of $\Phi_0(\mathbb{H})$.

In [16], the determinant of an operator $L \in \mathcal{A}(\mathbb{H})$ is defined as $\det L := \det L|_{\mathbb{X}}$, where $L|_{\mathbb{X}}$ is the restriction of L (as domain and as codomain) to any finite dimensional subspace $\mathbb{X}$ of $\mathbb{H}$ containing the image of $I - L$, with the understanding that $\det L|_{\mathbb{X}} = 1$ if $\mathbb{X} = \{0\}$.

Here are some fundamental properties of the determinant.

Remark 3.1. Let $L, L_1, L_2 \in \mathcal{A}(\mathbb{H})$ be given. Then,

- $\det L \neq 0$ if and only if L is invertible;
- $R \in \mathrm{Iso}(\mathbb{H}, \mathbb{K})$ implies $RLR^{-1} \in \mathcal{A}(\mathbb{K})$ and $\det(RLR^{-1}) = \det L$;
- $L_2 L_1 \in \mathcal{A}(\mathbb{H})$ and $\det(L_2 L_1) = \det L_2 \det L_1$.

See, for example, [9] for a discussion about other properties.

The easy proof of the following remark is left to the reader.

Remark 3.2. Let $\mathbb{X} \oplus \mathbb{Y}$ be a splitting of $\mathbb{H}$ with $\dim \mathbb{X} < \infty$. Suppose that, according to this splitting, $L \in \mathcal{L}(\mathbb{H})$ can be represented in a block matrix form as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ 0 & I_{22} \end{pmatrix},$$

where I_{22} is the identity operator on $\mathbb{Y}$. Then $L \in \mathcal{A}(\mathbb{H})$ and $\det L = \det L_{11}$.

Given $L \in \Phi_0(\mathbb{H}, \mathbb{K})$, in [6], an operator $K \in \mathcal{F}(\mathbb{H}, \mathbb{K})$ was called a *corrector of* L if $L + K \in \mathrm{Iso}(\mathbb{H}, \mathbb{K})$. Since the word "corrector" is misleading (an invertible operator need not to be corrected), we will use the more appropriate word *companion*.

Any $L \in \mathrm{Iso}(\mathbb{H}, \mathbb{K})$ has a *natural companion*: the trivial element of the space $\mathcal{L}(\mathbb{H}, \mathbb{K})$. This fact was of fundamental importance for two concepts of orientation introduced in [6] and, consequently, for the construction of the bf-degree.

Given any $L \in \Phi_0(\mathbb{H}, \mathbb{K})$, let $\mathcal{C}(L)$ denote the subset of the vector space $\mathcal{F}(\mathbb{H}, \mathbb{K})$ of the companions of L. Whatever is L, invertible or non-invertible, this set is nonempty and, according to the following definition, it admits a partition in exactly two equivalence classes (see [6] for details).

Definition 3.3 (*Equivalence relation of companions*). Two companions K_1 and K_2 of an operator $L \in \Phi_0(\mathbb{H}, \mathbb{K})$ are *equivalent* (more precisely, *L-equivalent)* if the determinant of the det-admissible operator $(L + K_2)^{-1}(L + K_1)$ is positive. This is an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes.

The following concept, introduced in [6], is based on Definition 3.3.

Definition 3.4 (*Algebraic orientation of a Φ_0-operator*). An *algebraic orientation* of an operator $L \in \Phi_0(\mathbb{H},\mathbb{K})$, for short called *alg-orientation*, is one of the two equivalence classes of $\mathcal{C}(L)$, denoted by $\mathcal{C}_+(L)$ and called the class of *positive companions* of the *alg-oriented operator* L. The two orientations of an operator $L \in \Phi_0(\mathbb{H},\mathbb{K})$ will be called *opposite (one to the other).*

Some special algebraic orientations are in order. Definition 3.6 deals with the finite dimensional case.

Definition 3.5 (*Natural alg-orientation of an isomorphism*). Any invertible operator $L\colon \mathbb{H} \to \mathbb{K}$ admits the *natural alg-orientation*: the one given by regarding the trivial operator of $\mathcal{L}(\mathbb{H},\mathbb{K})$ as a positive companion of L.

Definition 3.6 (*Associated alg-orientation of a linear operator*). Let $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension. Assume that they are oriented up to an inversion of both the orientations (or, equivalently, assume that $\mathbb{H}\times\mathbb{K}$ is oriented). Then any $L \in \mathcal{L}(\mathbb{H},\mathbb{K})$ admits the alg-orientation which is *associated to the orientations of* $\mathbb{H}$ *and* $\mathbb{K}$: the one given by considering as a positive companion of L any $K \in \mathcal{C}(L)$ such that $L + K$ is orientation preserving.

Recall that, if $\dim \mathbb{H} < \infty$ and $L \in \mathcal{L}(\mathbb{H})$, by $\operatorname{sign} L$ one simply means the sign of $\det L$. More generally, if $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension and are oriented, then the (classical) sign of L is defined as follows (see, for example, [19]):

$$\operatorname{sign} L = \begin{cases} 0 \text{ if } L \text{ is not invertible,} \\ +1 \text{ if } L \text{ is orientation preserving,} \\ -1 \text{ if } L \text{ is orientation reversing.} \end{cases}$$

Going beyond the finite dimensional context, we introduce the following concept of sign of an alg-oriented operator, called here *bf-sign* in order to distinguish it from the above classical notion.

Definition 3.7 (*bf-sign of an alg-oriented operator*). Let $L \in \Phi_0(\mathbb{H},\mathbb{K})$ be alg-oriented. Its *bf-sign* is the integer

$$\operatorname{sign}_{bf} L = \begin{cases} 0 \text{ if } L \text{ is not invertible,} \\ +1 \text{ if } L \text{ is invertible and naturally alg-oriented,} \\ -1 \text{ if } L \text{ is invertible and not naturally alg-oriented.} \end{cases}$$

The easy proof of the following remark is left to the reader.

Remark 3.8. Assume that $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension and are oriented, then, given $L \in L(\mathbb{H},\mathbb{K})$, one has $\operatorname{sign} L = \operatorname{sign}_{bf} L$, provided that L has the associated alg-orientation.

3.2. Topological orientation

We now extend the concept of orientation to nonlinear maps (see [6,7] for more details).

The basic fact is that the alg-orientation of an operator $\check{L} \in \Phi_0(\mathbb{H},\mathbb{K})$ induces an alg-orientation to the operators in a neighborhood of $\check{L}$. In fact, since $\operatorname{Iso}(\mathbb{H},\mathbb{K})$ and $\Phi_0(\mathbb{H},\mathbb{K})$ are open in $\mathcal{L}(\mathbb{H},\mathbb{K})$, any companion of $\check{L}$ remains a companion of all L sufficiently close to $\check{L}$.

Definition 3.9 (*Top-oriented maps of Φ_0-operators in the flat case*). Let $\mathcal{X}$ be a topological space and $\Gamma\colon \mathcal{X} \to \Phi_0(\mathbb{H},\mathbb{K})$ a continuous map. A function ω that to any $x \in \mathcal{X}$ assigns an alg-orientation $\omega(x)$ of $\Gamma(x)$ is called a *topological orientation* of Γ (*top-orientation* for short) provided it is *locally constant* in the following sense: if $\check{x} \in \mathcal{X}$ and $K \in \omega(\check{x})$, then $K \in \omega(x)$ for all x in a neighborhood of $\check{x}$. The map Γ is called *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen. A subset $\mathcal{A}$ of $\Phi_0(\mathbb{H},\mathbb{K})$ is *top-orientable* or *top-oriented* if so is the inclusion map $\mathcal{A} \hookrightarrow \Phi_0(\mathbb{H},\mathbb{K})$.

Definition 3.10 (*Pull-back of an orientation in the flat case*). Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. If $\sigma\colon \mathcal{X} \to \mathcal{Y}$ and $\Gamma\colon \mathcal{Y} \to \Phi_0(\mathbb{H},\mathbb{K})$ are continuous maps, then any orientation ω of Γ *induces* an orientation ω^* of the composite map $\Gamma \circ \sigma$ by putting $\omega^*(x) = \omega(\Gamma(\sigma(x)))$ for all $x \in \mathcal{X}$. We will say that the orientation ω^* is the *pull-back of* ω or, informally, that the orientation ω^* is *induced on* $\Gamma \circ \sigma$ *by* ω.

From Definition 3.10 one gets that $\Phi_0(\mathbb{H},\mathbb{K})$ is locally top-orientable and, if $\mathcal{A} \subseteq \Phi_0(\mathbb{H},\mathbb{K})$ is top-orientable, then so is any subset of $\mathcal{A}$, as it is any continuous map $\Gamma\colon \mathcal{X} \to \mathcal{A}$. In particular, any constant map from a topological space into $\Phi_0(\mathbb{H},\mathbb{K})$ is top-orientable.

If $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension, then $\Phi_0(\mathbb{H},\mathbb{K})$, which in this case coincides with $\mathcal{L}(\mathbb{H},\mathbb{K})$, is top-orientable. In fact, assuming that $\mathbb{H}$ and $\mathbb{K}$ are oriented, one can assign, according to Definition 3.6, the associated alg-orientation to any operator of $\mathcal{L}(\mathbb{H},\mathbb{K})$. One can check that this turns out to be a top-orientation.

However, $\Phi_0(\mathbb{H},\mathbb{K})$ could be not top-orientable in the infinite dimensional context. In fact, a surprising result of N. Kuiper (see [10,18]) asserts that, if $\mathbb{H}$ is infinite dimensional and separable, then the linear group $\mathrm{GL}(\mathbb{H})$ is contractible. Actually, in the context of Banach spaces, in [7, Theorem 3.15] it is shown that, given a Banach space $\mathbb{F}$, if $\mathrm{GL}(\mathbb{F})$ is connected, then the open subset $\Phi_0(\mathbb{F})$ of $\mathcal{L}(\mathbb{F})$ is as well connected but not top-orientable.

Hereafter, by $\hat{\Phi}_0(\mathbb{H},\mathbb{K})$ we shall mean the set of the alg-oriented Φ_0-operators acting from $\mathbb{H}$ to $\mathbb{K}$. Namely,

$$\hat{\Phi}_0(\mathbb{H},\mathbb{K}) = \big\{\big(L,\alpha\big) : L \in \Phi_0(\mathbb{H},\mathbb{K}),\ \alpha \text{ is an alg-orientation of } L\big\}.$$

Remark 3.11. (A two-fold covering space in the flat case) Definition 3.9 implies that the set $\hat{\Phi}_0(\mathbb{H},\mathbb{K})$ can be endowed with the topology which makes the natural projection

$$\mathcal{P}\colon \hat{\Phi}_0(\mathbb{H},\mathbb{K}) \to \Phi_0(\mathbb{H},\mathbb{K})$$

a 2-fold covering space (see [7] for details). Therefore, given a continuous map $\Gamma\colon \mathcal{X} \to \Phi_0(\mathbb{H},\mathbb{K})$ defined on a topological space, the second component ω of a lift

$$x \in \mathcal{X} \mapsto \big(\Gamma(x),\omega(x)\big) \in \hat{\Phi}_0(\mathbb{H},\mathbb{K})$$

of Γ is a top-orientation of Γ, and the function

$$\hat{\Gamma} := (\Gamma,\omega)\colon \mathcal{X} \to \hat{\Phi}_0(\mathbb{H},\mathbb{K})$$

may be regarded, abusing of the notation introduced in Definition 3.9, as a top-oriented map of Φ_0-operators.

The covering space $\mathcal{P}\colon \hat{\Phi}_0(\mathbb{H},\mathbb{K}) \to \Phi_0(\mathbb{H},\mathbb{K})$ is useful for dealing with the top-orientability, as well as with the top-orientation, of continuous maps into $\Phi_0(\mathbb{H},\mathbb{K})$; this is the case, for example, of the differential $df\colon \mathcal{X} \to \Phi_0(\mathbb{H},\mathbb{K})$ of a Φ_0-map $f\colon \mathcal{X} \to \mathbb{K}$ defined on an open subset of $\mathbb{H}$.

Since, by definition, a simply connected topological space is assumed to be path connected, from Remark 3.11 and the theory of covering spaces, one gets that, if $\mathcal{X}$ is simply connected and locally path connected, then $\Gamma\colon \mathcal{X} \to \Phi_0(\mathbb{H},\mathbb{K})$ admits exactly two top-orientations. Moreover, if $\check{x} \in \mathcal{X}$ and α is any of the two alg-orientations of $\Gamma(\check{x})$, then there exists one and only one top-orientation ω of Γ such that $\omega(\check{x}) = \alpha$.

Particular attention should be paid to the special convex subset $LS(\mathbb{H})$ of $\mathcal{L}(\mathbb{H})$ consisting of the compact linear perturbations of the identity, that we shall call *Leray–Schauder subset* of $\mathcal{L}(\mathbb{H})$. Notice that Property 5 of Proposition 2.1 implies that $LS(\mathbb{H})$ is contained in $\Phi_0(\mathbb{H})$. Therefore, since it is simply connected and locally path connected, the following definition makes sense.

Definition 3.12 (*Standard top-orientation of the Leray–Schauder subset of* $\mathcal{L}(\mathbb{H})$). The unique top-orientation ω of $LS(\mathbb{H})$ such that $\omega(I)$ is the natural alg-orientation of the identity (see Definition 3.5) will be called *standard*.

We now recall the concept of top-orientation for maps between manifolds. To this end, we need an analogous 2-fold covering space for the more general case in which the Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$ are replaced by two manifolds $\mathcal{M}$ and $\mathcal{N}$. For this purpose we define two sets: the *base space*

$$\Phi_0^+(\mathcal{M},\mathcal{N}) = \big\{(p,q,L) : p \in \mathcal{M}, q \in \mathcal{N}, L \in \Phi_0(T_p\mathcal{M}, T_q\mathcal{N})\big\};$$

and the *total space*

$$\hat{\Phi}_0^+(\mathcal{M},\mathcal{N}) = \big\{(p,q,L,\alpha) : (p,q,L) \in \Phi_0^+(\mathcal{M},\mathcal{N}),\ \alpha \text{ is an alg-orientation of } L\big\}.$$

We need to define the topologies on these two sets in order to make the *natural projection* $\mathcal{P}^+\colon (p,q,L,\alpha) \mapsto (p,q,L)$ a covering space.

The topology on the base space $\Phi_0^+(\mathcal{M},\mathcal{N})$ is defined as follows. Let $\varphi\colon U \to \mathbb{H}$ and $\psi\colon V \to \mathbb{K}$ be two charts of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then, there exists a bijective correspondence between the subset $\Phi_0^+(U,V)$ of $\Phi_0^+(\mathcal{M},\mathcal{N})$ and the topological space $\Phi_0^+(\varphi(U),\psi(V)) = \varphi(U) \times \psi(V) \times \Phi_0(\mathbb{H},\mathbb{K})$. In fact, one can check that the map $\mathcal{I}\colon \Phi_0^+(U,V) \to \Phi_0^+(\varphi(U),\psi(V))$, defined by

$$(p,q,L) \mapsto \big(\varphi(p), \psi(q), d\psi_q\, L\, (d\varphi_p)^{-1}\big),$$

is a bijection. Thus, in $\Phi_0^+(U,V)$ we consider the topology which makes $\mathcal{I}$ a homeomorphism; obtaining, in this way, a neighborhood basis of any element $(\check{p},\check{q},\check{L}) \in \Phi_0^+(U,V)$. Since φ and ψ are arbitrary, we get a topology on $\Phi_0^+(\mathcal{M},\mathcal{N})$.

Similarly, we define the topology on the total space $\hat{\Phi}_0^+(\mathcal{M},\mathcal{N})$. Let $\varphi\colon U\to\mathbb{H}$ and $\psi\colon V\to\mathbb{K}$ be two charts as above, and consider the bijection

$$\hat{\mathcal{I}}\colon \hat{\Phi}_0^+(U,V)\to\hat{\Phi}_0^+(\varphi(U),\psi(V)),$$

defined by $(p,q,L,\alpha)\mapsto\big(\varphi(p),\psi(q),\,d\psi_q\,L\,(d\varphi_p)^{-1},\,d\psi_q\,\alpha\,(d\varphi_p)^{-1}\big)$, where

$$d\psi_q\,\alpha\,(d\varphi_p)^{-1}=\big\{d\psi_q\,K\,(d\varphi_p)^{-1}:K\in\alpha\big\}.$$

Since $\hat{\Phi}_0^+(\varphi(U),\psi(V))=\varphi(U)\times\psi(V)\times\hat{\Phi}_0(\mathbb{H},\mathbb{K})$ is a topological space, the topology on $\hat{\Phi}_0^+(\mathcal{M},\mathcal{N})$ can be defined with exactly the same argument used for $\Phi_0^+(\mathcal{M},\mathcal{N})$, and this implies the following result.

Proposition 3.13. (The two-fold covering space in the non-flat case) *The above topologies on $\Phi_0^+(\mathcal{M},\mathcal{N})$ and $\hat{\Phi}_0^+(\mathcal{M},\mathcal{N})$ make the natural projection*

$$\mathcal{P}^+\colon\hat{\Phi}_0^+(\mathcal{M},\mathcal{N})\to\Phi_0^+(\mathcal{M},\mathcal{N}),\quad (p,q,L,\alpha)\mapsto(p,q,L)$$

a two-fold covering space.

Because of Proposition 3.13, the following notion of topological orientation of a map into $\Phi_0^+(\mathcal{M},\mathcal{N})$ makes sense.

Definition 3.14 (*Top-oriented maps of Φ_0-operators in the non-flat case*). Let $\mathcal{X}$ be a topological space and $\Gamma^+\colon\mathcal{X}\to\Phi_0^+(\mathcal{M},\mathcal{N})$ a continuous map. A *top-orientation* of Γ^+ is the second component ω of a lift

$$\hat{\Gamma}^+:=(\Gamma^+,\omega)\colon\mathcal{X}\to\hat{\Phi}_0^+(\mathcal{M},\mathcal{N})$$

of Γ^+. The map Γ^+ is called *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen. A subset $\mathcal{A}^+$ of $\Phi_0^+(\mathcal{M},\mathcal{N})$ is *top-orientable* or *top-oriented* if so is the inclusion map $\mathcal{A}^+\hookrightarrow\Phi_0^+(\mathcal{M},\mathcal{N})$.

The following orientability criterion can be deduced from the theory of covering spaces. Recall that a simply connected space is assumed to be path connected.

Proposition 3.15. (Orientability for maps of Φ_0-operators in the non-flat case) *Let $\Gamma^+\colon\mathcal{X}\to\Phi_0^+(\mathcal{M},\mathcal{N})$ be a continuous map defined on a topological space. If $\mathcal{X}$ is simply connected and locally path connected, then Γ^+ admits exactly two top-orientations. Moreover, if the restriction $\Gamma^+|_{\mathcal{E}}$ of Γ^+ to a path connected subset $\mathcal{E}$ of $\mathcal{X}$ has a top-orientation $\breve{\omega}$, then there exists a unique top-orientation $\hat{\omega}$ of Γ^+ whose restriction to $\mathcal{E}$ coincides with $\breve{\omega}$.*

Let us now introduce the notion of orientation for nonlinear Fredholm maps between Hilbert spaces. The following important definition is propaedeutic to the more important concept of orientation for nonlinear Fredholm maps between manifolds, that we will introduce in Definition 3.18 below.

Definition 3.16 (*Top-orientation of Φ_0-maps in the flat case*). Let W be an open subset of $\mathbb{H}$ and $f\colon W\to\mathbb{K}$ a Φ_0-map. A *top-orientation* of f is a top-orientation of the differential $df\colon W\to\Phi_0(\mathbb{H},\mathbb{K})$, according to Definition 3.9 or, equivalently, according to Remark 3.11. The map f is said to be *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen.

A special and important case of a Φ_0-map defined on an open subset W of $\mathbb{H}$ is a C^1 *compact vector field*. Namely, a C^1 map $\mathfrak{f}\colon W \to \mathbb{H}$ with the property that $k := I - \mathfrak{f}$ is a compact map; that is, a map sending bounded sets into relatively compact sets (see, e.g., [11, Sect. 8]). It is known that, for any $p \in W$, the differential $d\mathfrak{f}_p$ belongs to the convex subset $LS(\mathbb{H})$ of $\Phi_0(\mathbb{H})$ consisting of the compact linear perturbation of the identity operator I. Therefore, the following definition makes sense.

Definition 3.17 (*Standard top-orientation of a C^1 compact vector field*). Given an open subset W of $\mathbb{H}$ and a C^1 compact vector field $\mathfrak{f}$ on W, the *standard top-orientation of* $\mathfrak{f}$ is the one induced on $d\mathfrak{f}\colon W \to LS(\mathbb{H})$ by the standard top-orientation of $LS(\mathbb{H})$, according to Definitions 3.10 and 3.12.

An operator $L \in \Phi_0(\mathbb{H}, \mathbb{K})$ can also be regarded as a C^1 map from $W = \mathbb{H}$ into $\mathbb{K}$. Therefore, for L we have two different notions of orientation: the alg-orientation (see Definition 3.4) and the top-orientation (see Definition 3.16). Since $dL\colon \mathbb{H} \to \mathbb{K}$ is the constant map $dL_p = L$ for all $p \in \mathbb{H}$, we shall tacitly assume that the two possible orientations, if given, coincide. More precisely: if $\mathcal{C}_+(L)$ is the class of positive companions for L, then it is as well for dL_p for all $p \in \mathbb{H}$.

Definition 3.18 (*Top-oriented Φ_0-maps between manifolds*). A *top-orientation* of a Φ_0-map $f\colon \mathcal{M} \to \mathcal{N}$ between two manifolds is a top-orientation of the continuous map

$$df^+\colon \mathcal{M} \to \Phi_0^+(\mathcal{M}, \mathcal{N}), \quad p \mapsto (p, f(p), df_p),$$

according to Definition 3.14. The map f is *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been assigned.

The next result is a direct consequence of Definition 3.18 and Proposition 3.15. Recall that a simply connected space is assumed to be path connected and observe that any manifold is locally path connected.

Proposition 3.19. (Orientability criterion for Φ_0-maps between manifolds) *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a Φ_0-map between two manifolds. If $\mathcal{M}$ is simply connected, then f admits exactly two top-orientations.*

A simple example of a not top-orientable Φ_0-map is given by a constant map f from the 2-dimensional real projective space $\mathbb{P}^2$ into $\mathbb{R}^2$ (see [7]). This is due to the fact that even dimensional real projective spaces are non-orientable. Incidentally, we observe that, although f is constant, this is not the case for the continuous map

$$df^+\colon \mathbb{P}^2 \to \Phi_0^+(\mathbb{P}^2, \mathbb{R}^2), \quad p \mapsto (p, f(p), df_p).$$

Definition 3.20 (*Three special top-orientations for Φ_0-maps between manifolds*).

(Natural) A special top-orientation of a Φ_0-map between manifolds is the *natural* one, which makes sense whenever $f\colon \mathcal{M} \to \mathcal{N}$ is a diffeomorphism (or, more generally, a local diffeomorphism): given any $p \in \mathcal{M}$, according to Definition 3.5, one assigns the natural alg-orientation to the differential df_p.

(Associated) Given a C^1 map $f\colon \mathcal{M} \to \mathcal{N}$ between two finite dimensional oriented manifolds of the same dimension, one can assign to f the top-orientation which is *associated to the orientations of* $\mathcal{M}$ *and* $\mathcal{N}$: to any differential df_p one assigns the alg-orientation which is associated to the orientations of $T_p\mathcal{M}$ and $T_{f(p)}\mathcal{N}$, according to Definition 3.6.

(Canonical) If f is a self-map of a connected, orientable, finite dimensional manifold $\mathcal{M}$, one can assign to f the *canonical top-orientation*: the one which is associated to any orientation of $\mathcal{M}$, provided that it is the same as domain and as codomain of f.

Now we define the concept of Φ_0-*homotopy* between two manifolds, as well as the notion of its top-orientation.

Definition 3.21 (*Homotopy of* Φ_0-*maps between manifolds*). Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, a homotopy $\mathcal{H}\colon \mathcal{M}\times[0,1] \to \mathcal{N}$ is a Φ_0-*homotopy* if it is continuously differentiable with respect to the first variable, and such that, given any *parameter* $t \in [0,1]$, the *partial map* $\mathcal{H}_t := \mathcal{H}(\cdot, t)\colon \mathcal{M} \to \mathcal{N}$ is Fredholm of index 0. Any two partial maps of a Φ_0-homotopy are said to be Φ_0-*homotopic*.

Denoting by $\partial_1\mathcal{H}\colon (p,t) \mapsto d(\mathcal{H}_t)_p$ the partial differential of $\mathcal{H}$ with respect to the first variable, a *top-orientation* of $\mathcal{H}$ is, according to Definition 3.14, a top-orientation of the map

$$(\partial_1\mathcal{H})^+\colon \mathcal{M}\times[0,1] \to \Phi_0^+(\mathcal{M},\mathcal{N}), \quad (p,t) \mapsto (p, \mathcal{H}(p,t), d(\mathcal{H}_t)_p).$$

A Φ_0-homotopy $\mathcal{H}$ is said to be *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen.

Let ω be a top-orientation of a Φ_0-homotopy $\mathcal{H}$ from $\mathcal{M}$ to $\mathcal{N}$. Then, given any parameter $t \in [0,1]$, $\omega_t := \omega(\cdot, t)$ is a top-orientation of the partial map $\mathcal{H}_t$, called the *partial top-orientation of* $\mathcal{H}$ *at* t. Conversely, one has the following consequence of the theory of covering spaces (see [6,7]).

Proposition 3.22. (Transport of the top-orientations for Φ_0-maps) *Assume that a partial map* $\mathcal{H}_t\colon \mathcal{M} \to \mathcal{N}$ *of a* Φ_0*-homotopy* $\mathcal{H}$ *has a top-orientation* α*. Then, there exists one and only one top-orientation* ω *of* $\mathcal{H}$ *whose partial top-orientation* ω_t *coincides with* α*. In particular, if two maps from* $\mathcal{M}$ *to* $\mathcal{N}$ *are* Φ_0*-homotopic, then they are both top-orientable or both not top-orientable.*

From Proposition 3.22 we deduce that any self-map of a manifold $\mathcal{M}$ which is Φ_0-homotopic to the identity is top-orientable. Indeed, being a diffeomorphism, the identity admits the natural top-orientation (see Definition 3.20). This agrees with a well known fact: if a finite dimensional manifold is non-orientable, then it is not contractible.

Proposition 3.22 could be extended to a wider class than the Φ_0-homotopies: Φ_0-*families of maps between manifolds*, just by replacing the parameter space $[0,1]$ with a simply connected and locally path connected topological space.

4. Topological degree

Here we summarize the main concepts related to the degree introduced in [6] for maps between real Banach manifolds, hereafter called *bf-degree* to distinguish it from other classical degrees, such as the Brouwer degree, *Br-degree*, and the Leray–Schauder degree, *LS-degree* (see [5,7,8] for additional details).

By an axiomatic approach, as in the work of Amann–Weiss [3] regarding the uniqueness of the Leray–Schauder degree, in [8] it is shown that the bf-degree is the only possible integer valued function satisfying three fundamental properties called *Normalization, Additivity and Homotopy Invariance* (see below for precise statements).

To be more specific, we need to define the domain of the bf-degree function. Given a top-oriented Φ_0-map $f\colon \mathcal{M} \to \mathcal{N}$ between two manifolds, an open (possibly empty) subset U of $\mathcal{M}$, and a *target value* $q \in \mathcal{N}$, the triple (f, U, q) is said to be *bf-admissible* for the bf-degree if $U \cap f^{-1}(q)$ is compact. From the axiomatic point of view, the bf-degree is an integer valued function, $\deg_{bf}$, defined on the class of all the bf-admissible triples, that satisfies the following three *fundamental properties*:

- (Normalization) *If* $f\colon \mathcal{M} \to \mathcal{N}$ *is a naturally top-oriented diffeomorphism onto an open subset of* $\mathcal{N}$, *then*
$$\deg_{bf}(f, \mathcal{M}, q) = 1, \quad \forall q \in f(\mathcal{M}).$$
- (Additivity) *Let* (f, U, q) *be a bf-admissible triple. If* U_1 *and* U_2 *are two disjoint open subsets of* U *such that* $U \cap f^{-1}(q) \subseteq U_1 \cup U_2$, *then*
$$\deg_{bf}(f, U, q) = \deg_{bf}(f|_{U_1}, U_1, q) + \deg_{bf}(f|_{U_2}, U_2, q).$$
- (Homotopy Invariance) *Assume that* $\mathcal{H}\colon \mathcal{M}\times[0,1] \to \mathcal{N}$ *is a top-oriented* Φ_0-*homotopy and* $\sigma\colon [0,1] \to \mathcal{N}$ *is a continuous path. If the set*
$$\big\{(p,t) \in \mathcal{M}\times[0,1] : \mathcal{H}(p,t) = \sigma(t)\big\}$$
is compact, then $\deg_{bf}(\mathcal{H}(\cdot, t), \mathcal{M}, \sigma(t))$ does not depend on $t \in [0,1]$.

Remark 4.1. Notice that the above Homotopy Invariance Property applies whenever $\mathcal{H}$ is a proper map.

Other useful properties of the bf-degree can be deduced from the three fundamental ones (see [8] for details). One of them is the

- (Localization) *If* (f, U, q) *is a bf-admissible triple, then so is* $(f|_U, U, q)$ *and*
$$\deg_{bf}(f, U, q) = \deg_{bf}(f|_U, U, q).$$

Another one is the

- (Excision) *If* (f, U, q) *is bf-admissible and* V *is an open subset of* U *such that* $f^{-1}(q) \cap U \subseteq V$, *then*
$$\deg_{bf}(f, U, q) = \deg_{bf}(f, V, q).$$

A significant one is the

- (Existence) *If* (f, U, q) *is bf-admissible and* $\deg_{bf}(f, U, q) \neq 0$, *then the equation* $f(p) = q$ *admits at least one solution in* U.

In some sense, the integer $\deg_{bf}(f, U, q)$ is an algebraic count of the solutions in U of the equation $f(p) = q$. More precisely, as a consequence of the fundamental properties, one gets the

- (Computation Formula) *If* (f, U, q) *is bf-admissible and* q *is a regular value for* f *in* U, *then the set* $U \cap f^{-1}(q)$ *is finite and*

$$\deg_{bf}(f, U, q) \;=\; \sum_{p \in U \cap f^{-1}(q)} \operatorname{sign}_{bf} df_p, \tag{4.1}$$

with the convention that the sum is zero if $U \cap f^{-1}(q)$ *is empty.*

An important subclass of the bf-admissible triples is given by the elements (f, U, q) with the following additional properties:

- f is proper on the closure $\overline{U}$ of U (in $\mathcal{M}$);
- q belongs to the open subset $\mathcal{N} \backslash f(\partial U)$ of $\mathcal{N}$.

Observe that, if (f, U, q) satisfies the above two properties, then $f^{-1}(q) \cap \overline{U}$ is a compact set which does not intersect ∂U. Hereafter such a triple will be called *strictly bf-admissible.*

The following straightforward consequence of the Homotopy Invariance Property holds for the strictly bf-admissible triples.

- (Continuous Dependence) *Let* $f \colon \mathcal{M} \to \mathcal{N}$ *be a top-oriented* Φ_0-map *and* U *an open subset of* $\mathcal{M}$. *If* f *is proper on the closure of* U, *then the function* $\deg_{bf}(f, U, \cdot) \colon \mathcal{N} \backslash f(\partial U) \to \mathbb{Z}$ *is continuous (hence, locally constant).*

A simple example of a bf-admissible triple which is not strictly bf-admissible is given by $(\exp, \mathbb{R}, q)$, where $q \in \mathbb{R}$ is arbitrary and the function exp is assumed to be canonically top-oriented (see Definition 3.20); in fact, the integer valued function $\deg_{bf}(\exp, \mathbb{R}, \cdot)$ is discontinuous at $0 \in \mathbb{R}$.

Suppose that a strictly bf-admissible triple $(f, \mathcal{M}, q)$ satisfies the following additional property:

- the codomain $\mathcal{N}$ of f is connected.

Then, the above Continuous Dependence Property implies that $\deg_{bf}(f, \mathcal{M}, q)$ does not depend on the target $q \in \mathcal{N}$. Therefore, as for the Brouwer degree of a self-map of a compact, connected, orientable manifold, we will adopt a short notation.

Definition 4.2. Let $f \colon \mathcal{M} \to \mathcal{N}$ be a top-oriented Φ_0-map. The symbol $\deg_{bf}(f)$ stands for $\deg_{bf}(f, \mathcal{M}, q)$, with $q \in \mathcal{N}$ arbitrary, provided that f is proper and $\mathcal{N}$ is connected.

Notice that, in the above notation, the open set $\mathcal{M}$ in which the degree is considered is not mentioned. In fact, this is unnecessary since, in this case, $\mathcal{M}$ is the whole domain of the map f, and the domain of a function is included in its formal definition as a triple: domain, codomain and graph.

As a simple example of degree for strictly bf-admissible triples consider a complex polynomial P of (algebraic) degree $n > 0$ and regard it as a self-map of $\mathbb{R}^2$. Then P is a proper map and the Computation Formula (4.1) shows that $\deg_{bf}(P) = n$, provided that P is canonically top-oriented (see Definition 3.20).

Another important class of bf-admissible triples is given by the Leray–Schauder C^1-triples. Namely, triples $(\mathfrak{f}, U, q)$, in which U is a bounded open subset of $\mathbb{H}$, the map $\mathfrak{f}\colon \overline{U} \to \mathbb{H}$ is a compact vector field, $q \notin \mathfrak{f}(\partial U)$, the restriction $\mathfrak{f}|_U$ is C^1 and has the standard top-orientation (see Definition 3.17). As pointed out in [7], with such a top-orientation, $\deg_{bf}(\mathfrak{f}, U, q)$ is the same as $\deg_{LS}(\mathfrak{f}, U, q)$.

We close this section with the following result regarding Φ_0-maps in the finite dimensional context. Recall the notions, given in Definition 3.20, of associated and canonical top-orientations.

Proposition 4.3. (bf-degree versus Br-degree for maps between oriented manifolds of the same dimension) *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a C^1 map between two oriented manifolds of the same finite dimension. If f is proper on the closure of an open subset U of $\mathcal{M}$, then, given any $q \in \mathcal{N} \setminus f(\partial U)$, one has*

$$\deg_{bf}(f, U, q) = \deg_{Br}(f, U, q),$$

provided that the top-orientation of f is associated to the orientations of $\mathcal{M}$ and $\mathcal{N}$. In particular, if f is a C^1 self-map of a connected, orientable manifold, and f is canonically top-oriented and proper, then $\deg_{bf}(f) = \deg_{Br}(f)$.

Proof. Taking into account Sard's Lemma, it is a consequence of the definition of the Brouwer degree (see e.g. [19]), Remark 3.8, and the Computation Formula (4.1) of the bf-degree. □

5. Compact and finitely perturbed vector fields

Let $\mathbb{H}$, $\mathbb{B}$, $\mathbb{S}$ and $\mathfrak{r}\colon \mathbb{H}\setminus\{0\} \to \mathbb{S}$ be as before. From now onward we shall assume that the dimension of $\mathbb{H}$ is at least 3, so that the unit sphere $\mathbb{S}$ is simply connected.

Hereafter $\mathfrak{f} = I - k$ denotes a compact vector field on $\mathbb{H}$. Since $\mathfrak{f}$ is a compact perturbation of the identity, which is proper on bounded closed subsets of $\mathbb{H}$, $\mathfrak{f}$ inherits the same property. Therefore, the image under $\mathfrak{f}$ of a bounded and closed set is as well bounded and closed. In particular, so is $\mathfrak{f}(\mathbb{S})$.

Recall that the *boundary map* (of $\mathfrak{f}$) is the restriction $\mathfrak{f}|_{\mathbb{S}}\colon \mathbb{S} \to \mathbb{H}$ of $\mathfrak{f}$ to $\mathbb{S}$. Moreover, if $0 \notin \mathfrak{f}(\mathbb{S})$, it makes sense to define the *boundary self-map* $\mathfrak{f}^\partial\colon \mathbb{S} \to \mathbb{S}$ as the composition $\mathfrak{r} \circ \mathfrak{f}|_{\mathbb{S}}$.

By $\mathfrak{g}\colon \mathbb{H} \to \mathbb{H}$ we shall denote a *finitely perturbed vector field*. Namely, a compact vector field with the additional property that the image of the *perturbing map* $h = I - \mathfrak{g}$ is contained in a finite dimensional subspace of $\mathbb{H}$. Thus, if $\mathbb{H}$ is finite dimensional, any continuous self-map of $\mathbb{H}$ is a finitely perturbed vector field. If $\dim \mathbb{H} = \infty$, h must send bounded sets into bounded sets, since otherwise $\mathfrak{g}$ would not be a compact vector field.

5.1. Compact vector fields

The following crucial result asserts that the boundary self-map $\mathfrak{f}^\partial$, whenever it is defined, is a proper map.

Lemma 5.1. *If* $0 \notin \mathfrak{f}(\mathbb{S})$*, then the boundary self-map* $\mathfrak{f}^\partial$ *is proper.*

Proof. Observe that the composition of proper maps is a proper map. Nevertheless, $\mathfrak{f}^\partial = \mathfrak{r} \circ \mathfrak{f}|_{\mathbb{S}}$ is not the composition of two proper maps: $\mathfrak{f}|_{\mathbb{S}}$ is proper, but $\mathfrak{r}$ is not, being defined on its natural domain $\mathbb{H}\backslash\{0\}$, needed to ensure the composition of $\mathfrak{r}$ with any boundary map $\mathfrak{f}|_{\mathbb{S}}$ such that $0 \notin \mathfrak{f}(\mathbb{S})$.

However, taking into account that, given $\mathfrak{f}$, one has $\mathfrak{r}\circ\mathfrak{f}|_{\mathbb{S}} = \mathfrak{r}|_{\mathfrak{f}(\mathbb{S})}\circ\mathfrak{f}|_{\mathbb{S}}$, the assertion will follow if we prove that the restriction $\mathfrak{r}|_{\mathfrak{f}(\mathbb{S})}$ of $\mathfrak{r}$ to $\mathfrak{f}(\mathbb{S})$ is a proper map. To check this, recall that $\mathfrak{f}(\mathbb{S})$ is bounded and closed. Consequently, it is contained in a subset of $\mathbb{H}\backslash\{0\}$ of the type $A = [a, b]\mathbb{S}$, with $0 < a < b < \infty$. Hence, since $\mathfrak{f}(\mathbb{S})$ is closed, it is enough to show that $\mathfrak{r}$ is proper on A; and this is true since, given a compact subset C of $\mathbb{S}$, one has $(\mathfrak{r}|_A)^{-1}(C) = [a, b]C$, which is a compact set. □

Assume that the compact vector field $\mathfrak{f} = I - k$ is C^1. Then the differential $d\mathfrak{f}_p$ at any point $p \in \mathbb{H}$ is given by $d\mathfrak{f}_p(\dot{p}) = \dot{p} - dk_p(\dot{p})$, with $\dot{p} \in T_p\mathbb{H} = \mathbb{H}$. Recalling that the Fréchet differential at a point of a compact map is a compact linear operator, we get that $d\mathfrak{f}_p = I - dk_p$ is a compact linear perturbation of the identity and, consequently, a Φ_0-operator (see Property 5 of Proposition 2.1). This shows that $\mathfrak{f}$ is a Fredholm map of index 0. Therefore, since $\mathbb{S}$ is a submanifold of $\mathbb{H}$ of codimension 1, from Property 2 of Proposition 2.1, we get the following assertion regarding the boundary map $\mathfrak{f}|_{\mathbb{S}}$.

Remark 5.2. If $\mathfrak{f}$ is of class C^1, then $\mathfrak{f}|_{\mathbb{S}}$ is a Fredholm map of index -1.

Observe that the differential $d\mathfrak{r}_q$ at any $q \in \mathbb{H}\backslash\{0\}$ of the radial retraction

$$\mathfrak{r}\colon \mathbb{H}\backslash\{0\} \to \mathbb{S}, \quad q \mapsto q/\|q\|$$

is $\frac{1}{\|q\|}\Pi_{q^\perp}$, where $\Pi_{q^\perp}$ is the orthogonal projection of $\mathbb{H}$ onto the tangent space $T_{\mathfrak{r}(q)}\mathbb{S} = q^\perp$ of $\mathbb{S}$ at $\mathfrak{r}(q)$. Thus, $d\mathfrak{r}_q$ is a Fredholm linear operator of index 1 and its kernel is the 1-dimensional subspace $\mathbb{R}q$ of $\mathbb{H}$. Therefore, one gets the following property of the radial retraction.

Remark 5.3. The radial retraction $\mathfrak{r}\colon \mathbb{H}\backslash\{0\} \to \mathbb{S}$ is a Fredholm map of index 1.

Now we get an important property of the boundary self-map of $\mathfrak{f}$.

Lemma 5.4. *If* $\mathfrak{f}$ *is* C^1 *and* $0 \notin \mathfrak{f}(\mathbb{S})$*, then* $\mathfrak{f}^\partial$ *is Fredholm of index* 0*.*

Proof. The assertion follows from Property 2 of Proposition 2.1, since $\mathfrak{f}^\partial$ is the composition of two Fredholm maps: the boundary map $\mathfrak{f}|_{\mathbb{S}}$, whose index is -1 (see Remark 5.2), and the radial retraction $\mathfrak{r}$, whose index is 1 (see Remark 5.3). □

Let $\mathfrak{f}$ be C^1 and such that $0 \notin \mathfrak{f}(\mathbb{S})$. From Lemma 5.1 and Lemma 5.4 one gets that, if $\mathfrak{f}$ is top-oriented, then the integer $\deg_{bf}(\mathfrak{f}^\partial)$ is well defined (recall Definition 4.2). However, since $\mathfrak{f}^\partial$ admits exactly two opposite top-orientations (see Proposition 3.19), the absolute value of the bf-degree of $\mathfrak{f}^\partial$ makes sense no matter what top-orientation we choose. The following statement is therefore valid.

Remark 5.5. Let $\mathfrak{f}$ be C^1 and such that $0 \notin \mathfrak{f}(\mathbb{S})$. Then, $|\deg_{bf}(\mathfrak{f}^\partial)|$ is well defined.

5.2. Finitely perturbed vector fields

Let $\mathfrak{g}$ be a finitely perturbed vector field on $\mathbb{H}$ and let $\mathbb{X}$ be a finite dimensional subspace of $\mathbb{H}$ containing the image of $h = I - \mathfrak{g}$. Denoting $\mathbb{Y} = \mathbb{X}^\perp$, consider the splitting $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$. Taking into account of Remark 3.2, we obtain the following result.

Lemma 5.6. *Assume that $\mathfrak{g} = I - h$ is of class C^1. With the decomposition $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$, the differential $d\mathfrak{g}_p$ of $\mathfrak{g}$ at a given point $p \in \mathbb{H}$ can be represented in a block matrix form as follows:*

$$d\mathfrak{g}_p = \begin{pmatrix} d\mathfrak{g}_p|_\mathbb{X} & -dh_p|_\mathbb{Y} \\ 0 & I|_\mathbb{Y} \end{pmatrix},$$

where each of the entries is the restriction of a linear operator to a convenient subspace of $\mathbb{H}$. Consequently, $\det d\mathfrak{g}_p = \det d\mathfrak{g}_p|_\mathbb{X}$.

Proof. The differential $d\mathfrak{g}_p \colon T_p\mathbb{H} \to T_{\mathfrak{g}(p)}\mathbb{H}$ of $\mathfrak{g}$ at p is the operator of $\mathcal{L}(\mathbb{H})$ given by $(d\mathfrak{g}_p)\dot{p} = \dot{p} - (dh_p)\dot{p}$. Thus, putting $\dot{p} = \dot{x} + \dot{y}$, with $\dot{x} \in T_p\mathbb{X} = \mathbb{X}$ and $\dot{y} \in T_p\mathbb{Y} = \mathbb{Y}$, one gets

$$(d\mathfrak{g}_p)\dot{p} = (d\mathfrak{g}_p)\dot{x} + (d\mathfrak{g}_p)\dot{y} = (d\mathfrak{g}_p|_\mathbb{X})\dot{x} + \dot{y} - (dh_p|_\mathbb{Y})\dot{y}.$$

Thus, the claimed matrix representation of $d\mathfrak{g}_p$ follows from the fact that

$$(d\mathfrak{g}_p|_\mathbb{X})\dot{x} = \dot{x} - (dh_p|_\mathbb{X})\dot{x} \in \mathbb{X}, \quad \dot{y} \in \mathbb{Y} \quad \text{and} \quad (dh_p|_\mathbb{Y})\dot{y} \in \mathbb{X}.$$

Finally, Remark 3.2 yields the equality $\det d\mathfrak{g}_p = \det d\mathfrak{g}_p|_\mathbb{X}$. □

Hereafter the unit sphere $\mathbb{E} := \mathbb{X} \cap \mathbb{S}$ of $\mathbb{X}$ will be called *equatorial sphere* of $\mathbb{S}$. Given $p \in \mathbb{E}$, denote, as usual, by $T_p\mathbb{S}$ or $p^\perp$ the tangent space of $\mathbb{S}$ at p, and by $T_p\mathbb{E}$ the tangent space of $\mathbb{E}$ at p. The next remark presents three facts that will be useful in the sequel. Leaving the easy proof to the reader, we only stress that it is crucial in the third property the fact that $p \in \mathbb{X}$.

Remark 5.7. Given $p \in \mathbb{E}$, the following properties hold:

1. the map $\mathfrak{g}$ sends $\mathbb{X}$ into itself;
2. the radial retraction $\mathfrak{r}$ sends $\mathbb{X}\backslash\{0\}$ onto $\mathbb{E}$;
3. $T_p\mathbb{S} = (\mathbb{X} \cap p^\perp) \oplus \mathbb{Y} = T_p\mathbb{E} \oplus \mathbb{Y}$.

Properties 1 and 2 in Remark 5.7 justify the following definition.

Definition 5.8. If $\mathfrak{g}$ is C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$, then the restriction $\mathfrak{g}^\partial|_\mathbb{E}$ (as domain and as codomain) of the boundary self-map $\mathfrak{g}^\partial$ to the equatorial sphere $\mathbb{E}$ will be called the *equatorial self-map (of* $\mathfrak{g}$*)*.

Notice that, if $\mathfrak{g}$ is C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$, then $\mathfrak{g}^\partial|_\mathbb{E}$ is Fredholm of index 0, since its domain and codomain have the same finite dimension.

Lemma 5.9. *If $\mathfrak{g}$ is C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$, then a point of $\mathbb{E}$ is regular for the boundary self-map $\mathfrak{g}^\partial \colon \mathbb{S} \to \mathbb{S}$ if and only if so is for the equatorial self-map $\mathfrak{g}^\partial|_\mathbb{E} \colon \mathbb{E} \to \mathbb{E}$.*

Proof. Given $p \in \mathbb{E}$, put $q = \mathfrak{g}^\partial(p)$, and observe that, according to Remark 5.7, $q \in \mathbb{E}$. It is sufficient to prove that the differentials

$$d(\mathfrak{g}^\partial)_p \colon T_p\mathbb{S} \to T_q\mathbb{S} \quad \text{and} \quad d(\mathfrak{g}^\partial|_\mathbb{E})_p \colon T_p\mathbb{E} \to T_q\mathbb{E}$$

are both injective or both non-injective. In fact, the two linear operators, being Fredholm of index 0, are injective if and only if they are surjective (and when this holds, by definition, p is a regular point).

Observe that, if $\dot{x} \in T_p\mathbb{E}$, then $d(\mathfrak{g}^\partial)_p\dot{x}$ is the same as $d(\mathfrak{g}^\partial|_\mathbb{E})_p\dot{x}$. In addition, since $q \in \mathbb{X}$, as in Remark 5.7, we have the splitting

$$T_q\mathbb{S} = T_q\mathbb{E} \oplus Y.$$

Therefore, denoting $u = \mathfrak{g}(p)$, taking into account of Lemma 5.6 and putting $\dot{p} \in T_p\mathbb{S}$ in the form $\dot{p} = \dot{x} + \dot{y}$, with $\dot{x} \in T_p\mathbb{E}$ and $\dot{y} \in \mathbb{Y}$, we may write, in a block matrix form,

$$d(\mathfrak{g}^\partial)_p\dot{p} = d\mathfrak{r}_u d\mathfrak{g}_p \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} d(\mathfrak{g}^\partial|_\mathbb{E})_p & -d\mathfrak{r}_u dh_p|_\mathbb{Y} \\ 0 & I|_\mathbb{Y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix},$$

where $h = I - \mathfrak{g}$. Because of the triangular form of the above square matrix, it is immediate to conclude that $d(\mathfrak{g}^\partial)_p$ is injective if and only if so is $d(\mathfrak{g}^\partial|_\mathbb{E})_p$. □

Lemma 5.10. *Assume $\mathfrak{g}$ of class C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$. Then, given $q \in \mathbb{E}$, one has $(\mathfrak{g}^\partial)^{-1}(q) = (\mathfrak{g}^\partial|_\mathbb{E})^{-1}(q)$. Thus, q is a regular value for $\mathfrak{g}^\partial$ if and only if so is for the equatorial self-map $\mathfrak{g}^\partial|_\mathbb{E}$.*

Proof. According to the splitting $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$, let $u = x + y \in \mathbb{H}\backslash\{0\}$ and observe that $\mathfrak{r}(u) = x/\|u\| + y/\|u\|$. Therefore $\mathfrak{r}(u) \in \mathbb{E}$ (if and) **only if** $u \in \mathbb{X}\backslash\{0\}$, which implies $\mathfrak{r}^{-1}(q) \subset \mathbb{X}$. Consequently, the equality $(\mathfrak{g}^\partial)^{-1}(q) = (\mathfrak{g}^\partial|_\mathbb{E})^{-1}(q)$ holds since, given any $v \in \mathbb{H}$, one has $\mathfrak{g}(v) \in \mathbb{X}$ (if and) only if $v \in \mathbb{X}$. The last assertion of the statement now follows from Lemma 5.9. □

We conclude this section with an approximation result that will be useful in the proof of Theorem 6.3.

Lemma 5.11 (Uniform approximation by finitely perturbed vector fields). *Let $\mathfrak{f}$ be a compact vector field of class C^n defined on $\mathbb{H}$. Given $\varepsilon > 0$, $\mathfrak{f}$ can be uniformly ε-approximated on $\overline{\mathbb{B}}$ by a finitely perturbed vector field of the same class as $\mathfrak{f}$.*

Proof. Given $\varepsilon > 0$ and denoting $\mathfrak{f} = I - k$, let $\mathcal{F} \subset k(\overline{\mathbb{B}})$ be a finite ε-net of the relatively compact set $k(\overline{\mathbb{B}})$. Consider the orthogonal projection $\Pi_\mathbb{X}$ onto the space $\mathbb{X}$ spanned by $\mathcal{F}$. Since $\Pi_\mathbb{X}$ is C^∞, the finitely perturbed vector field $\mathfrak{g} = I - \Pi_\mathbb{X} \circ k$ is of class C^n. Moreover, one has $\|\mathfrak{g}(p) - \mathfrak{f}(p)\| = \|k(p) - \Pi_\mathbb{X}(k(p))\| < \varepsilon$ for all $p \in \overline{\mathbb{B}}$ (recall that $\Pi_\mathbb{X}(k(p))$ is the point of $\mathbb{X}$ closest to $k(p)$). □

6. Existence of zeros and Intermediate Value Theorem

Let $\mathbb{H}$, $\mathbb{X}$, $\mathbb{Y} = \mathbb{X}^{\perp}$, $\mathbb{B}$, $\mathbb{S}$ and $\mathbb{E} = \mathbb{X} \cap \mathbb{S}$ be as in the previous sections. Hereafter, we shall assume that the dimension of $\mathbb{X}$ is at least 2, so that the equatorial sphere $\mathbb{E}$ is connected. Recall that the dimension of $\mathbb{H}$ is at least 3.

In the finite dimensional case, a classical result on the existence of zeros for a boundary map $\mathfrak{f}|_{\mathbb{S}} \colon \mathbb{S} \to \mathbb{H}$ is due to L. Kronecker (see [12,17]). This result is based on the definition of an integer, the Kronecker index of $\mathfrak{f}|_{\mathbb{S}}$, here denoted by $I(\mathfrak{f}|_{\mathbb{S}})$, whose definition depends only on the restriction of $\mathfrak{f}$ to the sphere $\mathbb{S}$.

A modern redefinition of the Kronecker index of $\mathfrak{f}|_{\mathbb{S}}$ based on differential forms is given in [12], and the integer obtained in this way turns out to coincide with the Brouwer degree, $\deg_{Br}(\mathfrak{f}^{\partial})$, of the boundary self-map $\mathfrak{f}^{\partial}$.

Theorem 6.1 (Kronecker's theorem on the existence of zeros). *Let $\mathfrak{f} \colon \mathbb{H} \to \mathbb{H}$ be a C^1 map on a finite dimensional real Hilbert space. Assume $0 \notin \mathfrak{f}(\mathbb{S})$, where $\mathbb{S}$ is the unit sphere of $\mathbb{H}$. If the Kronecker index $I(\mathfrak{f}|_{\mathbb{S}})$ of the boundary map $\mathfrak{f}|_{\mathbb{S}}$ is different from zero, then $\mathfrak{f}$ vanishes somewhere in the open ball $\mathbb{B}$. Therefore, due to the equality $I(\mathfrak{f}|_{\mathbb{S}}) = \deg_{Br}(\mathfrak{f}^{\partial})$, the same holds if the Brouwer degree of $\mathfrak{f}^{\partial}$ is nonzero.*

Lemma 6.2 below is a preliminary infinite dimensional version of the Kronecker's result. Recall that, if a compact vector field $\mathfrak{f}$ is C^1 and $0 \notin \mathfrak{f}(\mathbb{S})$, then $|\deg_{bf}(\mathfrak{f}^{\partial})|$ is well defined (see Remark 5.5).

Lemma 6.2 (Existence of zeros for finitely perturbed vector fields). *Let $\mathfrak{g}$ be a C^1 finitely perturbed vector field on a real Hilbert space $\mathbb{H}$. Assume $0 \notin \mathfrak{g}(\mathbb{S})$. If $|\deg_{bf}(\mathfrak{g}^{\partial})| \neq 0$, then $\mathfrak{g}$ vanishes somewhere in $\mathbb{B}$.*

Proof. Let $\mathbb{X}$ be any finite dimensional subspace of $\mathbb{H}$ containing the image of the perturbing map $I - \mathfrak{g}$, and let $\mathbb{E} = \mathbb{X} \cap \mathbb{S}$ be the corresponding equatorial sphere.

Taking into account that $\mathbb{E}$ is the unit sphere of $\mathbb{X}$ and that $\mathfrak{g}^{\partial}|_{\mathbb{E}}$ is the same as $(\mathfrak{g}|_{\mathbb{X}})^{\partial}$, according to Theorem 6.1 it is sufficient to prove that $\deg_{Br}(\mathfrak{g}^{\partial}|_{\mathbb{E}}) \neq 0$.

Since $\mathfrak{g}^{\partial}|_{\mathbb{E}}$ is a self-map of an orientable, finite dimensional, connected manifold, we may assume that it is canonically top-oriented (see Definition 3.20). Therefore, from Proposition 4.3, it follows that $\deg_{Br}(\mathfrak{g}^{\partial}|_{\mathbb{E}}) = \deg_{bf}(\mathfrak{g}^{\partial}|_{\mathbb{E}})$. Hence, it is enough to show that

$$|\deg_{bf}(\mathfrak{g}^{\partial}|_{\mathbb{E}})| = |\deg_{bf}(\mathfrak{g}^{\partial})|.$$

Actually, we will prove that, with a suitable top-orientation of $\mathfrak{g}^{\partial}$, we get

$$\deg_{bf}(\mathfrak{g}^{\partial}|_{\mathbb{E}}) = \deg_{bf}(\mathfrak{g}^{\partial}). \tag{6.1}$$

Given any $p \in \mathbb{E}$, put $q = \mathfrak{g}(p)$. Recalling Property 3 in Remark 5.7, we have

$$T_p\mathbb{S} = T_p\mathbb{E} \oplus \mathbb{Y} \quad \text{and} \quad T_q\mathbb{S} = T_q\mathbb{E} \oplus \mathbb{Y}.$$

Notice that a linear operator $K \in \mathcal{L}(T_p\mathbb{E}, T_q\mathbb{E})$ is a companion of $d(\mathfrak{g}^\partial|_\mathbb{E})_p$ if and only if, according with the above splittings, the *corresponding operator*

$$\check{K} := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$$

is a companion of $d(\mathfrak{g}^\partial)_p$.

One can check that K_1 and K_2 are equivalent companions of $d(\mathfrak{g}^\partial|_\mathbb{E})_p$ if and only if the corresponding operators $\check{K}_1$ and $\check{K}_2$ are equivalent companions of $d(\mathfrak{g}^\partial)_p$.

Thus, calling ω the canonical top-orientation of the equatorial self-map $\mathfrak{g}^\partial|_\mathbb{E}$, we get an alg-orientation $\check{\omega}(p)$ of any differential $d(\mathfrak{g}^\partial)_p$, $p \in \mathbb{E}$, by choosing as a positive companion any $\check{K}$ such that K is a positive companion of $d(\mathfrak{g}^\partial|_\mathbb{E})_p$.

Since $\omega(p)$ depends continuously on $p \in \mathbb{E}$, one can check that the same property is inherited by the map $p \mapsto \check{\omega}(p)$, $p \in \mathbb{E}$. Applying Proposition 3.15, we get a unique top-orientation of $\mathfrak{g}^\partial$ whose restriction to $\mathbb{E}$ coincides with $\check{\omega}$.

Take now a regular value $q \in \mathbb{E}$ for both $\mathfrak{g}^\partial|_\mathbb{E}$ and $\mathfrak{g}^\partial$ (see Lemma 5.10) and consider any $p \in (\mathfrak{g}^\partial)^{-1}(q) = (\mathfrak{g}^\partial|_\mathbb{E})^{-1}(q)$. Since p is a regular point for both the maps $\mathfrak{g}^\partial$ and $\mathfrak{g}^\partial|_\mathbb{E}$, the null operators $0 \in \mathcal{L}(T_p\mathbb{E}, T_q\mathbb{E})$ and $\check{0} \in \mathcal{L}(T_p\mathbb{S}, T_q\mathbb{S})$ are companions of the differentials $d(\mathfrak{g}^\partial|_\mathbb{E})_p$ and $d(\mathfrak{g}^\partial)_p$, respectively. Because of the relation between the orientations ω and $\check{\omega}$, they are both positive or negative companions of the above differentials, which, consequently, have the same bf-signs (see Definition 3.7). Finally, equality (6.1) holds because of the Computation Formula (4.1) of the bf-degree. □

The following result shows that the thesis of Lemma 6.2 still holds for every compact vector field.

Theorem 6.3 (Existence of zeros via bf-degree). *Let $\mathfrak{f}$ be a C^1 compact vector field on $\mathbb{H}$ and assume $0 \notin \mathfrak{f}(\mathbb{S})$. If $|\deg_{bf}(\mathfrak{f}^\partial)| \neq 0$, then $\mathfrak{f}$ vanishes somewhere in $\mathbb{B}$.*

Proof. Lemma 5.11 ensures the existence of a sequence $\{\mathfrak{g}_n\}$ of C^1 finitely perturbed vector fields converging uniformly to $\mathfrak{f}$ in the closed disk $\overline{\mathbb{B}}$. As $\mathfrak{f}^\partial$ is a closed map and $0 \notin \mathfrak{f}(\mathbb{S})$, there exists a ball centered at 0 having empty intersection with $\mathfrak{f}(\mathbb{S})$. Therefore, without loss of generality, we may assume that $0 \notin \mathfrak{g}_n(\mathbb{S})$ for every n. Hence, every $\mathfrak{g}_n^\partial$ is well defined. We may also suppose that any $\mathfrak{g}_n^\partial$ is Φ_0-homotopic to $\mathfrak{f}^\partial$ via the proper map $\mathcal{H}_n\colon \mathbb{S}\times[0,1] \to \mathbb{S}$ defined by $\mathcal{H}_n(p,t) = (t\mathfrak{g}_n + (1-t)\mathfrak{f})^\partial(p)$.

Choose any of the two possible top-orientations of $\mathfrak{f}^\partial$; so that, because of Proposition 3.22, we get a top-orientation of any $\mathcal{H}_n$. Thus, by the Homotopy Invariance Property of the bf-degree, we obtain

$$\deg_{bf}(\mathfrak{g}_n^\partial) = \deg_{bf}(\mathfrak{f}^\partial) \neq 0, \quad \forall n \in \mathbb{N}.$$

Applying Lemma 6.2, we get a sequence $\{p_n\}$ in $\overline{\mathbb{B}}$ such that $\mathfrak{g}_n(p_n) = 0$ for any n. Thus, one has $\|\mathfrak{f}(p_n)\| = \|\mathfrak{f}(p_n) - \mathfrak{g}_n(p_n)\| \to 0$ as $n \to +\infty$. This implies that the image of the sequence $\{\mathfrak{f}(p_n)\}$ is contained in a compact subset of $\mathbb{H}$. Consequently, since $\mathfrak{f}$ is proper on $\overline{\mathbb{B}}$, the sequence $\{p_n\}$ is as well contained in a compact set. Therefore, we may assume that $\{p_n\}$ converges to a point $\bar{p} \in \overline{\mathbb{B}}$, which, necessarily, is such that $\mathfrak{f}(\bar{p}) = 0$. Finally, $\bar{p} \in \mathbb{B}$ since, by assumption, $0 \notin \mathfrak{f}(\mathbb{S})$. □

Contrary to the one-dimensional case, Theorem 6.5 below, which is our most general version of the Intermediate Value Theorem, cannot be solely deduced from a result on the existence of zeros. To overcome the difficulty, we need, in addition, the following consequence of the Homotopy Invariance Property of the bf-degree.

Lemma 6.4. *Given a C^1 compact vector field $\mathfrak{f}$ on $\mathbb{H}$, let $\mathcal{C}$ be a connected component of $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$. Then, the map $u \in \mathcal{C} \mapsto |\deg_{bf}((\mathfrak{f}-u)^\partial)|$ is constant.*

Proof. According to Remark 5.5, given any $u \in \mathcal{C}$, the integer $|\deg_{bf}((\mathfrak{f}-u)^\partial)|$ is well defined. We need to prove that, if z_0 and z_1 are in $\mathcal{C}$, then

$$|\deg_{bf}((\mathfrak{f}-z_0)^\partial)| = |\deg_{bf}((\mathfrak{f}-z_1)^\partial)|. \tag{6.2}$$

Observe that $\mathcal{C}$ is open, since so is $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$. Therefore, the connected set $\mathcal{C}$ is actually path connected and, consequently, there exists a C^1 path $\gamma\colon [0,1] \to \mathcal{C}$ joining z_0 with z_1. Consider the Φ_0-homotopy $\mathcal{H}\colon \mathbb{S}\times[0,1] \to \mathbb{S}$ defined by $(p,t) \mapsto (\mathfrak{f}-\gamma(t))^\partial(p)$, choose any of the two possible top-orientations of the partial map $\mathcal{H}_0 = (\mathfrak{f}-\gamma(0))^\partial$ and call it α. From Proposition 3.22 we get that there exists one and only one top-orientation ω of $\mathcal{H}$ whose partial top-orientation ω_0 coincides with α. Thus, $\deg_{bf}((\mathfrak{f}-\gamma(t))^\partial)$ is defined for any $t \in [0,1]$ and, taking into account that $\mathcal{H}$ is proper, from Remark 4.1 one gets

$$\deg_{bf}((\mathfrak{f}-z_0)^\partial) = \deg_{bf}((\mathfrak{f}-z_1)^\partial),$$

which implies formula (6.2). □

We are now ready to prove our main result.

Theorem 6.5 (Intermediate Value Theorem via bf-degree). *Let $\mathfrak{f}$ be a compact vector field of class C^1 on a real Hilbert space $\mathbb{H}$. Given $q \notin \mathfrak{f}(\mathbb{S})$, assume that $|\deg_{bf}((\mathfrak{f}-q)^\partial)| \neq 0$. Then, the connected component of $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$ containing q is a bounded open subset of $\mathfrak{f}(\mathbb{B})$.*

Proof. Denote by $\mathcal{C}$ the connected component of $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$ containing q. Since $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$ is open, so is the component $\mathcal{C}$. Thus, recalling that a compact vector field maps bounded sets into bounded sets, it remains to prove that $\mathcal{C}$ is contained in $\mathfrak{f}(\mathbb{B})$; which means that $\mathfrak{f}-u$ vanishes somewhere in $\mathbb{B}$, whatever is $u \in \mathcal{C}$. According to Theorem 6.3, it is sufficient to prove that $|\deg_{bf}((f-u)^\partial)| \neq 0$ for all $u \in \mathcal{C}$, and this follows from Lemma 6.4 and the assumption $|\deg_{bf}((\mathfrak{f}-q)^\partial)| \neq 0$. □

Let $\mathfrak{f}$ be a compact vector field of class C^1 on $\mathbb{H}$, and $p \in \mathbb{S}$ a given point such that $\mathfrak{f}(p) \neq 0$. As pointed out before, $d\mathfrak{r}_{\mathfrak{f}(p)}$ is a Fredholm linear operator of index 1 (see Remark 5.3) and its kernel is the 1-dimensional subspace $\mathbb{R}\mathfrak{f}(p)$ of $\mathbb{H}$. If, in addition, $0 \notin \mathfrak{f}(\mathbb{S})$, the point p is regular for $\mathfrak{f}^\partial$ if and only if the restriction of $d\mathfrak{f}_p$ to the tangent space $T_p\mathbb{S} = p^\perp$ of $\mathbb{S}$ at p is injective and $\mathfrak{f}(p) \notin d\mathfrak{f}_p(p^\perp)$. Therefore we have the following assertion.

Remark 6.6. Let $\mathfrak{f}$ be a C^1 compact vector field such that $0 \notin \mathfrak{f}(\mathbb{S})$. A point $p \in \mathbb{S}$ is regular for $\mathfrak{f}^\partial$ if and only if $d\mathfrak{f}_p(p^\perp) + \mathbb{R}\mathfrak{f}(p) = \mathbb{H}$.

The notion of transversality is well known in literature (see e.g., [14,15]). Here we recall the definition in the special case we are interested in.

Definition 6.7. Let $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ be a compact vector field of class C^1 on a real Hilbert space, and Λ a half-line with extreme $q \notin \mathfrak{f}(\mathbb{S})$ and tangent vector $v \in \mathbb{S}$ such that $q + v \in \Lambda$. We say that the boundary map $\mathfrak{f}|_\mathbb{S}$ *intersects transversally* Λ for $p \in \mathbb{S}$ if $\mathfrak{f}(p) \in \Lambda$ and $d\mathfrak{f}_p(p^\perp) + \mathbb{R}v = \mathbb{H}$. If this holds for any $p \in \mathbb{S}$ such that $\mathfrak{f}(p) \in \Lambda$, we will say that $\mathfrak{f}|_\mathbb{S}$ *intersects transversally* Λ or, by abuse of terminology, that the set $\mathfrak{f}(\mathbb{S})$ *intersects transversally* Λ.

Notice that, when $\mathfrak{f}(\mathbb{S})$ intersects transversally Λ at a value $\mathfrak{f}(p)$, one has the transverse intersection of two C^1 manifolds: the image $\mathfrak{f}(U)$ of a convenient neighborhood U of p in $\mathbb{S}$ with the open half-line Λ. In fact, the transversality condition $d\mathfrak{f}_p(p^\perp) + \mathbb{R}v = \mathbb{H}$ of Definition 6.7 implies that $d\mathfrak{f}_p(p^\perp)$ has codimension 1 in $\mathbb{H}$; therefore, since $d\mathfrak{f}_p$ is Fredholm of index 0, $d(\mathfrak{f}|_\mathbb{S})_p$ must be injective, and this implies that the restriction of $\mathfrak{f}$ to a convenient neighborhood U of p in $\mathbb{S}$ is a C^1 diffeomorphism onto the C^1 manifold $\mathfrak{f}(U)$ of codimension 1 in $\mathbb{H}$. Observe also that the tangent space of $\mathfrak{f}(U)$ at $\mathfrak{f}(p)$ is $d\mathfrak{f}_p(p^\perp)$.

Lemma 6.8 (bf-degree via a half-line). *Let $\mathfrak{f}$ be a C^1 compact vector field on $\mathbb{H}$. Assume $0 \notin \mathfrak{f}(\mathbb{S})$. If the intersection of $\mathfrak{f}(\mathbb{S})$ with a half-line Λ with extreme 0 is transverse and its preimage under $\mathfrak{f}|_\mathbb{S}$ is made up of an odd number of points, then $|\deg_{bf}(\mathfrak{f}^\partial)| \neq 0$.*

Proof. Consider the singleton $\{q\} = \mathbb{S} \cap \Lambda$ and observe that the set $(\mathfrak{f}^\partial)^{-1}(q)$ is the same as $\{p \in \mathbb{S} : \mathfrak{f}(p) \in \Lambda\}$. According to Remark 6.6, this set contains only regular points of $\mathfrak{f}^\partial$. Therefore, given any of the two possible orientations of $\mathfrak{f}^\partial$, to evaluate $\deg_{bf}(\mathfrak{f}^\partial)$ we may apply the Computation Formula (4.1) with q as a regular value for $\mathfrak{f}^\partial$. In this formula, any point of $(\mathfrak{f}^\partial)^{-1}(q)$ contributes with -1 or 1. Consequently, since the cardinality of $(\mathfrak{f}^\partial)^{-1}(q)$ is odd, the algebraic count of these points cannot be zero. □

The next result is our easily understandable version of the Intermediate Value Theorem. It is a consequence of our main result, Theorem 6.5, but it is not equivalent, as shown in Example 6.10 below.

Theorem 6.9 (Intermediate Value Theorem via a half-line). *Let $\mathfrak{f}$ be a compact vector field of class C^1 on $\mathbb{H}$. Given $q \notin \mathfrak{f}(\mathbb{S})$, let Λ_q be a half-line with extreme q. If the intersection of $\mathfrak{f}(\mathbb{S})$ with Λ_q is transverse and its preimage under $\mathfrak{f}|_\mathbb{S}$ is made up of an odd number of points, then the connected component of $\mathbb{H}\backslash\mathfrak{f}(\mathbb{S})$ containing q is a bounded open subset of $\mathfrak{f}(\mathbb{B})$.*

Proof. Let $\Lambda = \Lambda_q - q$ be the half-line, with extreme 0, parallel to Λ_q. Observe that $(\mathfrak{f} - q)(\mathbb{S})$ intersects transversally Λ at the same points as $\mathfrak{f}(\mathbb{S})$ meets Λ_q. Thus, applying Lemma 6.8 to the map $\mathfrak{f} - q$, we get $|\deg_{bf}((\mathfrak{f} - q)^\partial)| \neq 0$, and the assertion follows from Theorem 6.5. □

The example below shows that, given any real Hilbert space $\mathbb{H}$ and any $n \in \mathbb{Z}$, there exists a top-oriented compact vector field $\mathfrak{f}_n$ on $\mathbb{H}$ such that $\deg_{bf}(\mathfrak{f}_n^\partial) = n$. Hence, if $n \neq 0$, as a consequence of Theorem 6.5, any value in the connected component of $\mathbb{H}\backslash\mathfrak{f}_n(\mathbb{S})$ containing $0 \in \mathbb{H}$ is assumed by $\mathfrak{f}_n$ in the open unit ball $\mathbb{B}$ of $\mathbb{H}$. In particular, such a component is bounded, since so is $\mathfrak{f}_n(\mathbb{B})$. As we shall see, Theorem 6.9 does not apply if the integer n is even. See also [1] for a similar example and the related discussion.

Example 6.10. Let S^1 denote the unit circle of the complex plane $\mathbb{C}$. Given any $n \in \mathbb{Z}$, consider the self-map σ_n of S^1 defined by $z \mapsto z^n$. It is well known, and easy to check, that the winding number (or, equivalently, the Brouwer degree) of this self-map is n.

If $n < 0$, the function σ_n admits an extension to the whole complex plane $\mathbb{C}$ given by $z \mapsto \bar{z}^{|n|}$, where, as usual, $\bar{z}$ denotes the conjugate of z. Therefore, given any n, we define the extension $\zeta_n \colon \mathbb{C} \to \mathbb{C}$ of σ_n by

$$\zeta_n(z) = \begin{cases} z^n & \text{if } n \geq 0, \\ \bar{z}^{|n|} & \text{if } n < 0. \end{cases}$$

Choose a 2-dimensional subspace $\mathbb{X}$ of $\mathbb{H}$ and identify it with the complex plane $\mathbb{C}$ by means of an isometric linear isomorphism, so that, for any n, the map ζ_n may be regarded as a self-map of $\mathbb{X}$ sending the unit circle $\mathbb{E}$ of $\mathbb{X}$ into itself (actually, onto itself, if $n \neq 0$). Recall the assumption $\dim \mathbb{H} > 2$ and put $\mathbb{Y} = \mathbb{X}^\perp$, so that $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$, with $\mathbb{Y}$ non-trivial.

Let $\psi \colon \mathbb{H} \to \mathbb{H}$ be any compact map of class C^1 with the following properties:

- $\psi(x) = 0$ for all $x \in \mathbb{X}$;
- for any $x \in \mathbb{X}$ the differential $d\psi_x$ sends $\mathbb{Y}$ into $\mathbb{X}$.

Then, the self-map $\mathfrak{f}_n$ of $\mathbb{H}$, defined by

$$\mathfrak{f}_n(x + y) = \zeta_n(x) + y - \psi(x + y), \quad x \in \mathbb{X},\ y \in \mathbb{Y},$$

is a compact vector field (actually, a finitely perturbed vector field if the second property is replaced by $\psi(\mathbb{H}) \subseteq \mathbb{X}$). Let us prove that, if we assume $0 \notin \mathfrak{f}_n(\mathbb{S})$, as it is verified in the special case $\psi(\mathbb{H}) \subseteq \mathbb{X}$, then $\deg_{bf}(\mathfrak{f}_n^\partial) = n$.

Observe that the restriction $\mathfrak{f}_n|_{\mathbb{E}}$ of $\mathfrak{f}_n$ to the equator $\mathbb{E} = S^1$ is a self-map of S^1. Therefore, it coincides with the equatorial self-map $\mathfrak{f}_n^\partial|_{\mathbb{E}}$, which is the same as the function $\sigma_n \colon S^1 \to S^1$ defined above, whose winding number is n.

Notice that, if $n = 0$, the connected component of $\mathbb{H}\backslash\mathfrak{f}_0(\mathbb{S})$ is unbounded. In fact, the pinched equatorial plane $\mathbb{C}\backslash\{1\}$ is contained in such a component.

Assume $n \neq 0$. With an analogous argument to that used in the proof of Lemma 6.2, one gets that, with one of the two possible top-orientations of $\mathfrak{f}_n^\partial$, we have $\deg_{bf}(\mathfrak{f}_n^\partial) = n$. Therefore, from Theorem 6.5 we get the following assertion:

Any value in the connected component of $\mathbb{H}\backslash\mathfrak{f}_n(\mathbb{S})$ containing the origin is assumed by $\mathfrak{f}_n$ in $\mathbb{B}$. Consequently, such a component is bounded.

Given an even integer $n \neq 0$, let $\mathfrak{f}_n$ and $\sigma_n = \mathfrak{f}_n|_{\mathbb{E}}$ be as in the Example 6.10. Let us show that Theorem 6.9 does not apply in this case. Consider a half-line Λ, with extreme 0, lying in the equatorial plane $\mathbb{X}$. This half-line meets transversally the equator $\mathbb{E} = S^1$ at only one value of σ_n, corresponding to an even number of points of S^1. By means of the intersection theory (see, for example, [2]), one gets that the intersection of $\mathfrak{f}_n(\mathbb{S})$ with any other half-line with extreme 0, if it is transverse, must be the image of an even number of points of $\mathbb{S}$. Therefore, the hypothesis of Theorem 6.9 is not satisfied whatever is the half line starting from the origin of $\mathbb{H}$.

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References

[1] Abbondandolo, A., Rot, T.O.: On the homotopy classification of proper Fredholm maps into a Hilbert space. J. Reine Angew. Math. **759**, 161–200 (2020)

[2] Abraham, R., Robbin, J.: Transversal mappings and flows. W.A. Benjamin Inc., New York (1967)

[3] Amann, H., Weiss, S.A.: On the uniqueness of the topological degree. Math. Z. **130**, 39–54 (1973)

[4] Benevieri, P., Calamai, A., Furi, M.: On the degree for oriented quasi-Fredholm maps: its uniqueness and its effective extension of the Leray–Schauder degree. Topol. Methods Nonlinear Anal. **46**(1), 401–430 (2015)

[5] Benevieri, P., Calamai, A., Furi, M., Pera, M.P.: A degree associated to linear eigenvalue problems in Hilbert spaces and applications to nonlinear spectral theory. J. Dynam. Differ. Equ. **34**(1), 555–581 (2022)

[6] Benevieri, P., Furi, M.: A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory. Ann. Sci. Math. Québec **22**, 131–148 (1998)

[7] Benevieri, P., Furi, M.: On the concept of orientability for Fredholm maps between real Banach manifolds. Topol. Methods Nonlinear Anal. **16**, 279–306 (2000)

[8] Benevieri, P., Furi, M.: On the uniqueness of the degree for nonlinear Fredholm maps of index zero between Banach manifolds. Commun. Appl. Anal. **15**, 203–216 (2011)

[9] Benevieri, P., Furi, M., Pera, M.P., Spadini, M.: About the sign of oriented Fredholm operators between Banach spaces. Z. Anal. Anwendungen **22**(3), 619–645 (2003)

[10] Bleecker, D., Booß-Bavnek, B.: Index Theory–with Applications to Mathematics and Physics. International Press, Somerville (2013)

[11] Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985)

[12] Dinca, G., Mawhin, J.: Brouwer degree. The core of nonlinear analysis, Progress in Nonlinear Differential Equations and Their Applications (PNLDE, volume 95) Birkhäuser, Cham (2021)

[13] Fitzpatrick, M.P., Pejsachowicz, J., Rabier, P.J.: Orientability of Fredholm families and topological degree for orientable nonlinear Fredholm mappings. J. Funct. Anal. **124**, 1–39 (1994)

[14] Guillemin, V., Pollack, A.: Differential Topology, Prentice-Hall Inc., Englewood Cliffs, New Jersey (1974)

[15] Hirsch, M.W.: Differential Topology, Graduate Texts in Math, vol. 33. Springer, Berlin (1976)

[16] Kato, T.: Perturbation Theory for Linear Operators, Classics in Mathematics. Springer-Verlag, Berlin (1995)

[17] Kronecker, L.: Ueber Systeme von Funktionen mehrerer Variabeln, Monatsber. Königl. Preussige Akademie Wiss. Berlin, Erste Abhandlung, 150–193; Zweite Abhandlung, 688–698 (1869)

[18] Kuiper, N.H.: The homotopy type of the unitary group of a Hilbert space. Topology **3**, 19–30 (1965)

[19] Milnor, J.W.: Topology from the Differentiable Viewpoint. University of Virginia Press, Charlottesville (1965)

[20] Outerelo, E., Ruiz. J.M.: Mapping degree theory, Graduate Studies in Mathematics, **108**, American Mathematical Soc., Providence, RI; Real Soc. Matemática Española, Madrid (2009)

[21] Pejsachowicz, J.: Orientation of Fredholm maps. J. Fixed Point Theory Appl. **2**, 97–116 (2007)

[22] Pejsachowicz, J., Rabier, P.J.: Degree theory for C^1-Fredholm mappings of index 0. J. Anal. Math. **76**, 289–319 (1998)

[23] Smale, S.: An infinite dimensional version of Sard's theorem. Am. J. Math. **87**, 861–866 (1965)

[24] Taylor, A.E., Lay, D.C.: Introduction to Functional Analysis. Wiley, New York-Chichester-Brisbane (1980)

[25] Väth, M.: Topological Analysis: From the Basics to the Triple Degree for Nonlinear Fredholm Inclusions. de Gruyter, Berlin (2012)

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